# 2.5a global stability

Monday, February 1, 2021 5:11 AM

**Def. 2.6** Suppose $\bar{x}$ is an equilibrium of the difference eq. $x_{t+1} = f(x_t)$, where $f: [0, a) \to [0, a)$, $0 < a \le \infty$. Then $\bar{x}$ is said to be **globally attractive** if $\forall\ x_0 \in (0, a)$, $\lim_{t \to \infty} x_t = \bar{x}$. The equilibrium $\bar{x}$ is **globally asymptotically stable** if it is globally attractive and locally stable.

**Aside:** If $f$ is continuous, then globally attracting implies locally asymptotically stable.

**Thm 2.5** If $f: [0, a) \to [0, a)$ is continuous, and $0 < f(x) < x$ for all $x \in (0, a)$, then the origin is globally asymptotically stable.

**proof.** $\{x_0, x_1, \dots\}$ is monotone decreasing and bounded by 0, and 0 is also the only fixed pt by supposition. □

**Thm 2.6** Let all of the following conditions hold:

(1) $f$ is a continuous function on $[0, a)$, $0 < a \le \infty$

(2) $f: [0, a) \to [0, a)$, $0 < a \le \infty$

(3) $f(0) = 0$, $f(\bar{x}) = \bar{x}$.

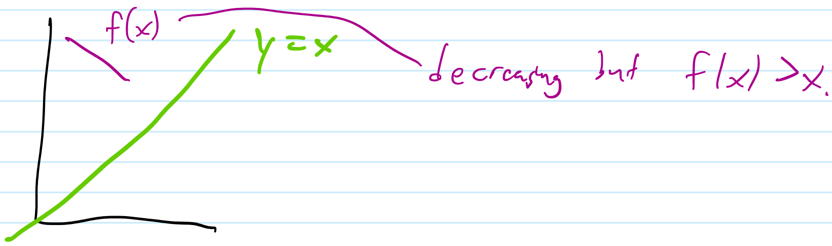
(4) $f(x) > x$ for $0 < x < \bar{x}$

(5) $f(x) < x$ for $\bar{x} < x < a$

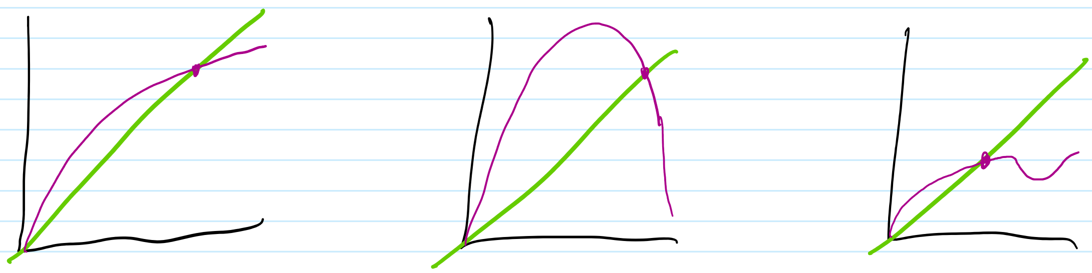
(6) If $\exists\ x_M \in (0, \bar{x})$ s.t. $f(x_M) = \sup_{x \in (0, \bar{x})} f(x)$, then $f$ is decreasing for $x > x_M$. ($x_M$ is a maximum in $(0, \bar{x})$)

When all conditions (1)–(6) hold, then $x_{t+1} = f(x_t)$ has a globally asymptotically stable equilibrium at $\bar{x}$ iff $f$ has no 2-cycles.

**Important note:** $f$ decreasing is NOT equiv. to $f(x) < x$




**Ex.**



**Thm 2.7** Let $f': I \to I$ be continuous. If $1 + f'(x) \neq 0 \ \forall x \in I$, then $x_{t+1} = f(x_t)$ has no 2-cycles in $I$.

**proof.** Suppose $\exists$ 2-cycle $\{x_0, x_1\}$ s.t. $f^2(x_0) = f(x_1) = x_0$, $x_0, x_1 \in I$.
WLOG, $x_0 < x_1$.

Then
$$\int_{x_0}^{x_1} (1 + f'(x))\, dx = (x_1 + f(x_1)) - (x_0 + f(x_0))$$
$$= f(x_0) + x_0 - x_0 - f(x_0) = 0.$$

Then by the MVT for integrals, $\exists c \in I$ s.t. $1 + f'(c) = 0$.

This is a contradiction. Hence, there are no 2-cycles in $I$.

**Thm 2.8** If $f: [0, a) \to [0, a)$ cont. and $\bar{x} \in (0, a)$ such that
$$x < f(x) < \bar{x} \quad \text{for} \quad 0 < x < \bar{x}$$
and $\bar{x} < f(x) < x$ for $x > \bar{x}$,
then the difference eq. $x_{t+1} = f(x_t)$ has a globally asymptotically stable equilibrium at $\bar{x}$.

**proof.** Note $x_t < x_{t+1} = f(x_t) < \bar{x}$ for $x_t < \bar{x}$

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and $\bar{x} < x_{t+1} = f(x_t) < x_t$ for $x_t > \bar{x}$

Case 1: $x_0 < \bar{x}$. Then $\{f^t(x_0)\}_{t=0}^{\infty}$ is monotone increasing and bounded above by $\bar{x}$,

Let $z_1 = \lim_{t\to\infty} f^t(x_0) = f\left(\lim_{t\to\infty} f^{t-1}(x_0)\right) = f(z_1)$, a fixed pt.

Case 2: $x_0 > \bar{x}$. Then $\{f^t(x_0)\}_{t=0}^{\infty}$ is monotone decreasing and bounded below by $\bar{x}$

Let $z_2 = \lim_{t\to\infty} f^t(x_0) = f(z_2)$, a fixed pt

But, the pos. fixed pt is $\bar{x}$, so $z_1 = z_2 = \bar{x}$. ☐

**Thm 2.9** Let $x_{t+1} = f(x_t)$

(1) $f$ is a continuous function on $[0, a)$, $0 < a \le \infty$
(2) $f : [0, a) \to [0, a)$, $0 < a \le \infty$
(3) $f(0) = 0$, $f(\bar{x}) = \bar{x}$.
(4) $f(x) > x$ for $0 < x < \bar{x}$
(5) $f(x) < x$ for $\bar{x} < x < a$
(6) If $\exists\, x_M \in (0, \bar{x})$ s.t. $f(x_M) = \sup_{x\in(0,\bar{x})} f(x)$, then $f$ is decreasing for $x > x_M$.

($x_M$ is a maximum in $(0, \bar{x})$)

(a) Suppose $f$ satisfies assumptions (1)–(5) but $f$ has no maximum in $(0, \bar{x})$. Then $\bar{x}$ is globally asymptotically stable.

(b) Suppose $f$ satisfies assumptions (1)–(6) and $f$ has a maximum $x_M$ in $(0, \bar{x})$. Then $\bar{x}$ is globally asymptotically stable iff $f(f(x)) > x$ for all $x \in [x_M, \bar{x})$.